More on Generalized Heisenberg Ferromagnet Models

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ABSTRACT

We generalize the integrable Heisenberg ferromagnet model according to each Hermitian symmetric spaces and address various new aspects of the generalized model. Using the first order formalism of generalized spins which are defined on the coadjoint orbits of arbitrary groups, we construct a Lagrangian of the generalized model from which we obtain the Hamiltonian structure explicitly in the case of $CP(N-1)$ orbit. The gauge equivalence between the generalized Heisenberg ferromagnet and the nonlinear Schrödinger models is given. Using the equivalence, we find infinitely many conserved integrals of both models.

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In the past decades, there have been extensive investigations on the structure of the continuous Heisenberg ferromagnet (HM) model [1]-[4]. The dynamical variable of the conventional HM model is given by a spin variable Q which is defined on the coadjoint orbit S^2 of $SU(2)$, i.e. $Q(x,t) = Q^a(x,t)T^a$, $\sum_{a=1}^{3} Q^a Q^a = k^2$, where T^a's are generators of the $SU(2)$ algebra and k^2 is a constant. This $SU(2)$ spin HM model was later extended to the $SU(N)$ case[5]. More generally, it was shown that there exists an extension of the HM model to each Hermitian symmetric spaces[6]. However, the extension in [6] is made only implicitly in connection with the nonlinear Schrödinger(NS) model. In particular, the integrability structure was not clear and the Lax pair formalism was lacking.

The purpose of this Letter is to provide a systematic understanding of the generalized HM model. We formulate the model in terms of a Lagrangian using the first order formalism of generalized spins which are defined on the coadjoint orbits of arbitrary groups. The gauge equivalence of the generalized HM and the generalized NS model is demonstrated. Especially, starting from the associated linear equation of the HM model, we obtain a closed form of the generalized NS equation(Eq.(17)). We also find zero curvature expressions of both the HM and the NS equations in terms of which we obtain infinitely many conserved integrals. These conserved integrals are constructed systematically by making use of the properties of Hermitian symmetric space and they are given in a multicomponent form thus giving more than "one series" of integrals. As an explicit example of the Lagrangian description, we perform a reduction to $CP(N-1)$ orbit in detail and explain the resulting Hamiltonian structure of the HM model.

We begin with a brief introduction on the Hermitian symmetric space. A symmetric space is a coset space G/K for Lie groups $G \supset K$ whose associated Lie algebras $\mathbf{g}$ and $\mathbf{k}$, with the decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$, satisfy the commutation relations,

$$[\mathbf{k}, \ \mathbf{k}] \subset \mathbf{k}, \ \ [\mathbf{k}, \ \mathbf{m}] \subset \mathbf{m}, \ \ [\mathbf{m}, \ \mathbf{m}] \subset \mathbf{k}. \tag{1}$$

A Hermitian symmetric space is a symmetric space equipped with a complex structure. For our purpose, we need only the following properties of Hermitian symmetric spaces [6][7] ; for each Hermitian symmetric space, there exists an element T in the Cartan subalgebra of $\mathbf{g}$ whose centralizer in $\mathbf{g}$ is $\mathbf{k}$, i.e. $\mathbf{k} = \{V \in \mathbf{g} : \ [V, \ T] = 0\}$. Also, up to a scaling, $J = \mathrm{ad}T = [T, \ *]$ is a linear map $J : \mathbf{m} \to \mathbf{m}$ satisfying the complex structure condition

$J^2 = \alpha$ for a constant α, or $[T, [T, M]] = \alpha M$, for $M \in \mathbf{m}$.

Now, we introduce an action for the HM model,

$$A = \int dt dx \ \mathrm{Tr} \ [2Tg^{-1}\dot{g} + \partial_x(gTg^{-1})\partial_x(gTg^{-1}) - 2BgTg^{-1}] \qquad (2)$$

where g is a map $g : R^2 \to G$ and $B = B(t)$ is an arbitrary element in $\mathbf{g}$ describing an external magnetic field. The equation of motion can be written in terms of a generalized spin $Q \equiv gTg^{-1}$,

$$\dot{Q} + \partial_x[Q, \ \partial Q] + [Q, \ B] = 0. \qquad (3)$$

The integrability of the HM equation (3) arises from the existence of its associated linear equations:

$$(\bar{\partial} - B - \lambda[Q, \ \partial Q] + \alpha\lambda^2 Q)\Psi_{HM} = 0, \quad (\partial + \lambda Q)\Psi_{HM} = 0, \qquad (4)$$

where $\partial = \partial/\partial x$, $\bar{\partial} = \partial/\partial t$ and λ is an arbitrary complex constant whereas α is a constant to be fixed later. These linear equations are overdetermined systems whose consistency requires the integrability condition:

$$
\begin{aligned}
0 &= [\bar{\partial} - B - \lambda[Q, \ \partial Q] + \alpha\lambda^2 Q, \ \partial + \lambda Q] \\
&= \lambda(\bar{\partial}Q + \partial[Q, \ \partial Q] + [Q, \ B]) + \lambda^2(-\alpha\partial Q + [Q, \ [Q, \ \partial Q]]).
\end{aligned} \qquad (5)
$$

The λ^1-order term in the second line of Eq.(5) becomes precisely the HM equation since the λ^2-order term vanishes identically due to the complex structure property of T,

$$
\begin{aligned}
[Q, \ [Q, \ \partial Q]] &= g[T, \ [T, \ g^{-1}(\partial Q)g]]g^{-1} = g[T, \ [T, \ [g^{-1}\partial g, \ T]]]g^{-1} \\
&= \alpha g[g^{-1}\partial g, \ T]g^{-1} = \alpha\partial Q,
\end{aligned} \qquad (6)
$$

or, whenever T satisfies a stronger condition:

$$T^2 = \beta T + \gamma I, \quad \alpha = \beta^2 + 4\gamma. \qquad (7)$$

Without loss of generality, we may set β to zero by shifting T by $T \to T - \beta/2$. This stronger condition holds at least for $SU(p+q)/S(U(p) \times U(q))$, $SO(2n)/U(n)$, $Sp(n)/U(n)$ compact Hermitian symmetric spaces and their noncompact counterparts. Since K is a subgroup of G commuting with T, this shows that Q in the case of Eq.(7) is in fact defined on the

coadjoint orbit G/K characterized by a number $Q^2 = \gamma I$. Note that the external magnetic field $B(t)$ can be made disappear by taking the gauge transformation of g and Ψ_{HM} such that $g \to b^{-1}(t)g$, $\Psi_{HM} \to b^{-1}(t)\Psi_{HM}$ where $b(t)$ satisfies $\bar{\partial}bb^{-1} = B(t)$. From now on, we assume that $B = 0$.

Having found the associated linear equations of the HM equation, we demonstrate the integrability of the HM model itself by deriving infinitely many conserved currents from the linear equations. In order to do so, we first derive the generalized NS equation from the generalized HM equation thereby proving the gauge equivalence of both models. Define $\Psi_{NS} \equiv g^{-1}\Psi_{HM}$ and rewrite the linear equation (4) in an equivalent form:

$$(\bar{\partial} + g^{-1}\bar{\partial}g - \lambda g^{-1}\partial g - \lambda^2 T)\Psi_{NS} = 0, \quad (\partial + g^{-1}\partial g + \lambda T)\Psi_{NS} = 0, \qquad (8)$$

where we have taken $\alpha = -1$ without loss of generality. Since Q is invariant under $g \to gk$ for $k \in K$, we choose k such that $g^{-1}\partial g$ is valued in $\mathbf{m}$. The integrability of Eq.(8) becomes the zero curvature condition:

$$[\bar{\partial} + g^{-1}\bar{\partial}g - \lambda g^{-1}\partial g - \lambda^2 T, \; \partial + g^{-1}\partial g + \lambda T] = 0. \qquad (9)$$

Equivalently, we may require

$$[T, \; g^{-1}\bar{\partial}g] - \partial(g^{-1}\partial g) = 0 \qquad (10)$$

and the identity

$$[\bar{\partial} + g^{-1}\bar{\partial}g, \; \partial + g^{-1}\partial g] = 0. \qquad (11)$$

$g^{-1}\bar{\partial}g$ may be expressed in terms of $g^{-1}\partial g$ by solving Eqs.(10) and (11) as follows; introduce a decomposition $g^{-1}\bar{\partial}g = (g^{-1}\bar{\partial}g)_m + (g^{-1}\bar{\partial}g)_k$ where subscript m and k refer to the components of $g^{-1}\bar{\partial}g$ in those vector subspaces. Then, due to the algebraic properties of Eq.(1), Eq.(10) becomes

$$[T, \; (g^{-1}\bar{\partial}g)_m] - \partial(g^{-1}\partial g) = 0 \qquad (12)$$

which can be solved for $(g^{-1}\bar{\partial}g)_m$ by applying the adjoint action of T,

$$[T, \; [T, \; (g^{-1}\bar{\partial}g)_m] = -(g^{-1}\bar{\partial}g)_m = [T, \; \partial(g^{-1}\partial g)]. \qquad (13)$$

4

The k-component of $g^{-1}\bar{\partial}g$ can be obtained from Eq.(11) which decomposes into

$$0 = \partial(g^{-1}\bar{\partial}g)_m - \bar{\partial}(g^{-1}\partial g) + [g^{-1}\partial g, \ (g^{-1}\bar{\partial}g)_k] \tag{14}$$

and

$$
\begin{aligned}
0 &= \partial(g^{-1}\bar{\partial}g)_k + [g^{-1}\partial g, \ (g^{-1}\bar{\partial}g)_m] \\
&= \partial(g^{-1}\bar{\partial}g)_k - [g^{-1}\partial g, \ [T, \ \partial(g^{-1}\partial g)]] \\
&= \partial(g^{-1}\bar{\partial}g)_k + \frac{1}{2}\partial[g^{-1}\partial g, \ [g^{-1}\partial g, \ T]].
\end{aligned} \tag{15}
$$

This may be integrated directly to yield

$$(g^{-1}\bar{\partial}g)_k = \frac{1}{2}[g^{-1}\partial g, \ [T, \ g^{-1}\partial g]] + C(t) \tag{16}$$

where the arbitrary function $C(t)$ can be set to zero by redefining g. Finally, the remaining equation (14) becomes the generalized NS equation:

$$\bar{\partial}(g^{-1}\partial g) + [T, \ \partial^2(g^{-1}\partial g)] + \frac{1}{2}[g^{-1}\partial g, \ [g^{-1}\partial g, \ [g^{-1}\partial g, \ T]]] = 0. \tag{17}$$

Thus we have shown the gauge equivalence of the HM equation and the NS equation.

Such a gauge equivalence also relates the conserved integrals of both models. In order to find the conserved integrals of the NS model, we first rewrite the linear equation in an equivalent form with $\Phi = \Psi_{NS}\exp(\lambda Tx - \lambda^2 Tt)$,

$$\bar{\partial}\Phi + (g^{-1}\bar{\partial}g - \lambda g^{-1}\partial g)\Phi - \lambda^2[T, \ \Phi] = 0, \quad \partial\Phi + g^{-1}\partial g\Phi + \lambda[T, \ \Phi] = 0, \tag{18}$$

which we solve iteratively by assuming

$$\Phi = \sum_{l=0}^{\infty} \frac{1}{\lambda^l}(\Phi_m^l + \Phi_k^l). \tag{19}$$

The subscript denotes the decomposition of Φ with the properties

$$[T, \ \Phi_k] = 0, \ \Phi_k\Phi_k \subset \Phi_k, \ \Phi_k\Phi_m \subset \Phi_m, \ \Phi_m\Phi_m \subset \Phi_k. \tag{20}$$

Explicit construction of such a matrix decomposition can be carried out for each Hermitian symmetric spaces given in [6]. For example, in the case of $SU(p+q)/S(U(p) \times Uq)$, we

note that any $U(p+q)$ group element Φ can be expressed as a sum of $\Phi_m = \sum c_i M^i$ and $\Phi_k = d_0 I + \sum d_j K^j$, where I is the identity matrix, M^i's are basis vectors for the symmetric space $SU(p+q)/S(U(p) \times U(q))$ and K^j's are the generators of the subalgebra $\mathbf{k}$. Obviously, this decomposition satisfies the relation Eq.(20). With Eq.(20), Eq.(18) changes into a set of recursive relations,

$$\partial \Phi_m^l + g^{-1}\partial g \Phi_k^l = -[T, \ \Phi_m^{l+1}] \tag{21}$$

$$\partial \Phi_k^l + g^{-1}\partial g \Phi_m^l = 0 \tag{22}$$

and

$$\bar{\partial} \Phi_m^l + (g^{-1}\bar{\partial} g)_m \Phi_k^l + (g^{-1}\bar{\partial} g)_k \Phi_m^l - g^{-1}\partial g \Phi_k^{l+1} = [T, \ \Phi_m^{l+2}] \tag{23}$$

$$\bar{\partial} \Phi_k^l + (g^{-1}\bar{\partial} g)_m \Phi_m^l + (g^{-1}\bar{\partial} g)_k \Phi_k^l - g^{-1}\partial g \Phi_m^{l+1} = 0. \tag{24}$$

Eq.(24) may be rewritten by using Eqs.(13), (16) and (21),

$$\bar{\partial} \Phi_k^l - [T, \ \partial(g^{-1}\partial g)]\Phi_m^l - g^{-1}\partial g[T, \ \partial \Phi_m^l] = 0. \tag{25}$$

Eqs.(21),(22) and (25) may be solved for Φ^l in lower order terms of iteration,

$$\begin{aligned}
\Phi_m^l &= [T, \ \partial \Phi_m^{l-1} + g^{-1}\partial g \Phi_k^{l-1}] \\
\Phi_k^l &= -\int dx g^{-1}\partial g \Phi_m^l + \int dt([T, \ \partial(g^{-1}\partial g)]\Phi_m^l + g^{-1}\partial g[T, \ \partial \Phi_m^l]).
\end{aligned} \tag{26}$$

Then, from the compatibility of Eqs.(22) and (25) ($\partial \bar{\partial} \Phi_k^l = \bar{\partial} \partial \Phi_k^l$), we obtain infinitely many conserved currents,

$$\bar{\partial} J_x^l + \partial J_t^l = 0 \ ; \quad l = 0, 1, 2, \cdots \tag{27}$$

where

$$\begin{aligned}
J_x^l &= -\partial \Phi_k^l = g^{-1}\partial g \Phi_m^l \\
J_t^l &= \bar{\partial} \Phi_k^l = [T, \ \partial(g^{-1}\partial g)]\Phi_m^l + g^{-1}\partial g[T, \ \partial \Phi_m^l].
\end{aligned} \tag{28}$$

In particular, if we choose an initial condition $\Phi_m^0 = 0$, $\Phi_k^0 = 1$, we obtain for $l = 1$,

$$J_x^1 = g^{-1}\partial g[T, \ g^{-1}\partial g], \quad J_t^1 = [T, \ \partial(g^{-1}\partial g)][T, \ g^{-1}\partial g] - g^{-1}\partial g \partial(g^{-1}\partial g). \tag{29}$$

6

In the simplest case where $G/K = SU(2)/U(1)$ or $SU(1,1)/U(1)$, we may take $g^{-1}\partial g$ and T by

$$g^{-1}\partial g = \begin{pmatrix} 0 & \kappa\psi \\ \kappa\psi^* & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix} \tag{30}$$

where $\kappa = 1$ for the compact $SU(2)/U(1)$ case and $\kappa = i$ for the noncompact $SU(1,1)/U(1)$ case. Then Eq.(17) reduces to the conventional nonlinear Schrödinger equation

$$\bar{\partial}\psi + i\partial^2\psi - 2i\kappa^2|\psi|^2\psi = 0. \tag{31}$$

In particular, the T-component of the conserved current in Eq.(29), i.e

$$
\begin{aligned}
0 &= \operatorname{Tr} T(\bar{\partial} J_x^1 + \partial J_t^1) \\
&= \bar{\partial}(\psi\psi^*) + \partial(i\psi^*\partial\psi - i\psi\partial\psi^*) \tag{32}
\end{aligned}
$$

corresponds to the probability conservation. Note that our conservation laws are given in a matrix form thus resulting more than one series of conserved local integrals. This is consistent with our generalized NS model as a multicomponent system.

The conserved integrals of the HM model can be derived from those of the NS model by using the gauge equivalence. In order to do so, we rewrite the iterative solution (26) in terms of $\Sigma_m^l \equiv g\Phi_m^l g^{-1}$ and $\Sigma_k^l \equiv g\Phi_k^l g^{-1}$ such that

$$
\begin{aligned}
\Sigma_m^l &= [Q, \ \partial\Sigma_m^{l-1} - [[Q, \ \partial Q], \Sigma_m^{l-1}]] - \partial Q \Sigma_k^{l-1} \\
\Sigma_k^l &= -\int dx\, J_x^l + \int dt\, J_t^l, \tag{33}
\end{aligned}
$$

where the conserved currents J_x^l, J_t^l are given by

$$
\begin{aligned}
J_x^l &\equiv -\partial\Sigma_k^l = -[[Q, \partial Q], \ \Sigma_k^l] + [Q, \partial Q]\Sigma_m^l \\
J_t^l &\equiv \bar{\partial}\Sigma_k^l = [\partial^2 Q + \frac{1}{2}[\partial Q, \ [Q, \ \partial Q]], \ \Sigma_k^l] - (\partial^2 Q + [\partial Q, \ [Q, \ \partial Q]])\Sigma_m^l \\
&\quad + [Q, \partial Q][Q, \ \partial\Sigma_m^l - [[Q, \partial Q], \ \Sigma_m^l]]. \tag{34}
\end{aligned}
$$

Obviously, these integrals, even though there are infinitely many, do not exhaust all conserved integrals of the HM model. For instance, the HM equation itself represents a conservation law $\bar{\partial}Q + \partial[Q, \ \partial Q] = 0$ but this does not arise from the above construction. This lacking is due to the specific choice of iterative solutions as in Eq.(19) and can be avoided by taking a different kind of iterative solutions.

For the rest of the Letter, we assume the stronger condition (7) and present, as an example of Lagrangian description of the HM model, an explicit reduction to the $CP(N-1)$ orbit. Let us first consider T given by

$$T = \text{diag}(t_1, \cdots, t_N), \quad t_1 = \cdots = t_n = a, \quad t_{n+1} = \cdots = t_N = b \tag{35}$$

with the traceless condition $na + (N - n)b = 0$. T satisfies the stronger condition (7) with $\beta = [(2n - N)/(n - N)]a$, $\gamma = a^2 n/(N - n)$. The isotropy group K for this element T is given by $K = SU(n) \times SU(N - n) \times U(1)$ and the coadjoint orbit is the Grassmannian manifold $SU(N)/(SU(n) \times SU(N - n) \times U(1))$. For the non-compact case, it corresponds to $SU(n, N - n)/(SU(n) \times SU(N - n) \times U(1))$. For other cases of Hermitian symmetric spaces, we may take an $2n \times 2n$ matrix T by

$$T = \begin{pmatrix} 0 & -t & \cdot & \cdot & \cdot & 0 \\ t & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & \ddots & & & \\ 0 & & & & 0 & -t \\ 0 & & & & t & 0 \end{pmatrix}. \tag{36}$$

so that $\beta = 0, \gamma = -t^2$. Then, the isotropy group for the above T is $U(n)$ and the corresponding integrable coadjoint orbits are given by [7] $SO(2n)/U(n)$, $Sp(2n)/U(n)$ and their noncompact counterparts. The remaining Hermitian symmetric spaces [6, 7] do not satisfy the stronger condition.

Now, we restrict to the $CP(N - 1) = SU(N)/(SU(N - 1) \times U(1))$ case. In this case, $T = i\text{diag}(N - 1, -1, \cdots, -1)$ and let us parametrize the group element g of $SU(N)$ by an N-tuple $g = (Z_1, Z_2, \cdots, Z_N)$, $Z_p \in \mathbf{C}^N$ $(p, q = 1, \cdots, N)$ such that

$$\bar{Z}_p Z_q = \delta_{pq}, \quad \det(Z_1, Z_2, \cdots, Z_N) = 1. \tag{37}$$

With this parametrization, the action (2) can be written in terms of Z,

$$A = \int dt dx \left[2Ni\bar{Z}_1 \dot{Z}_1 - 2N^2(\partial\bar{Z}_1 \partial Z_1 - (\bar{Z}_1 \partial Z_1)(\partial\bar{Z}_1 Z_1)) + \lambda(\bar{Z}_1 Z_1 - 1) \right] \tag{38}$$

where we imposed the constraint of Eq.(37) by introducing a Lagrangian multiplier λ. Note that all the Z_p's with $p = 2, \cdots, N$ have been eliminated and they do not appear in the subsequent analysis. The above action is invariant under the local $U(1)$ action, $Z_1 \rightarrow$

$e^{i\alpha(x,t)}Z_1, \bar{Z}_1 \to e^{-i\alpha(x,t)}\bar{Z}_1$, and the constraint $\bar{Z}_1 Z_1 - 1 \approx 0$ is a first class constraint. Let $Z_1^T \equiv (z_1, z_2, \cdots, z_N)$. Then, the constraint can be solved explicitly in a real gauge where $z_N = z_N^*$ $(z_N \neq 0)$ [8] in terms of $\psi_i = z_i/z_N (i = 1, 2, \cdots, N-1)$ so that

$$z_i = \frac{\psi_i}{\sqrt{1 + |\psi|^2}}, \quad z_N = \frac{1}{\sqrt{1 + |\psi|^2}}. \tag{39}$$

Substituting the above expression into the action (38), we obtain a reduced action on the $CP(N-1)$ orbit (up to a total derivative term),

$$A = \int dt dx \left(2iN \frac{\bar{\psi}_i \dot{\psi}_i}{1 + |\psi|^2} - 2N^2 g_{ij} \partial \psi_i \partial \bar{\psi}_j \right), \tag{40}$$

where g_{ij} is the Fubini-Study metric on $CP(N-1)$,

$$g_{ij} = \frac{(1 + |\psi|^2)\delta_{ij} - \bar{\psi}_i \psi_j}{(1 + |\psi|^2)^2}. \tag{41}$$

Note that the first term in the above action can be written as $2N \int dx \theta$ where θ is the canonical one-form on $CP(N-1)$ defined by $\theta = i\partial_\psi \log(1 + |\psi|^2)d\psi$. The classical dynamics of the above action can be described by a generalized Hamiltonian dynamics [9, 10] with the Hamiltonian given by

$$H = 2N^2 \int dx g_{ij} \partial \psi_i \partial \bar{\psi}_j. \tag{42}$$

The Poisson bracket is defined by the inverse matrix $\omega^{ij} = -ig^{ij}$ of the symplectic two-form $\omega = d\theta = w_{ij} d\psi_i d\bar{\psi}_j$ such that

$$\{F(\bar{\psi}, \psi), G(\bar{\psi}, \psi)\} = (2iN)^{-1} \int dx \, g^{ij} \left(\frac{\delta F}{\delta \bar{\psi}_i(x)} \frac{\delta G}{\delta \psi_j(x)} - \frac{\delta G}{\delta \bar{\psi}_i(x)} \frac{\delta F}{\delta \psi_j(x)} \right) \tag{43}$$

with the inverse Fubini-Study metric $g^{ij} = (1 + |\psi|^2)(\delta_{ij} + \bar{\psi}_i \psi_j)$. A simple calculation gives

$$\{\bar{\psi}_i(x,t), \psi_j(x',t)\} = (2iN)^{-1} g^{ij} \delta(x - x'), \quad \{\bar{\psi}_i(x,t), \bar{\psi}_j(x',t)\} = \{\psi_i(x,t), \psi_j(x',t)\} = 0. \tag{44}$$

The Hamiltonian description of the above model in terms of a generalized $SU(N)$ spin is more efficient because of the $SU(N)$ invariance. Consider the unreduced Q,

$$Q = iN Z_1 \bar{Z}_1 - iI. \tag{45}$$

Defining the generalized spin functions by $Q^a = 2\text{Tr}(QT^a)$ in which T^a's are generators satisfying the commutation relation $[T^a, T^b] = f^{abc}T^c$ and the normalization $\text{Tr}(T^aT^b) = (-1/2)\delta_{ab}$, we have

$$Q^a(\psi, \bar{\psi}) = 2iN \sum_{p,q=1}^{N} \bar{z}_p(T^a)_{pq}z_q \tag{46}$$

where Eq.(39) is assumed. A straightforward computation using Eq.(43) gives

$$\{Q^a(x,t), Q^b(x',t)\} = -f^{abc}Q^c(x,t)\delta(x-x'). \tag{47}$$

Also, using the Hamiltonian expressed in terms of Q^a's, $H = (1/2)\int dx(\partial Q^a)^2$, we recover the equation of motion (3) with $B = 0$:

$$\dot{Q}^a = \{H, Q^a\} = f^{abc}Q^b\partial^2 Q^c. \tag{48}$$

In this Letter, we have shown that the HM model and the NS model can be generalized according to each Hermitian symmetric spaces and their integrability structures can be studied systematically by making use of the properties of Hermitian symmetric space. These approach may be extended to the study of other properties of the generalized model, e.g. construction of soliton solutions and their scattering behaviors both classical and quantum. Also, our Lagrangian of the HM model might be used in quantizing the model through the path integral approach. These works are in progress and will be reported elsewhere [11].

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